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THE STRONG QUANTUM LIMIT OF THE RELAXATION TIME AND SHEAR

VISCOSITY OF A GAS OF LOADED SPHERE MOLECULES

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page 13, Eq. 2-27: Insert $[$ just after $\int$.

page 20, Eq. 3-12: Place bars above four J's in the denominator.

page 37, Eq. 4-34: The second bracket should read,

$$\left[\frac{52}{15} F_4 - \frac{52}{15} x F_3 + \frac{11}{15} x^2 F_2 + \frac{2}{45} x^3 F_1\right] .$$

THE STRONG QUANTUM LIMIT OF THE RELAXATION TIME AND SHEAR VISCOSITY OF A GAS OF LOADED SPHERE MOLECULES*

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ABSTRACT

Quantum mechanical expressions for the relaxation time and shear viscosity have been derived by Mueller for a gas consisting of loaded sphere molecules. These expressions are expanded in power series in $\frac{1}{\chi^2}$ where $\chi = \frac{\hbar}{\sigma\sqrt{2\mu kT}}$, i.e. in the strong quantum region. The coefficients of the first four terms for the reciprocal of the relaxation time are evaluated numerically.

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SECTION I

INTRODUCTION

Quantum mechanical expressions for the reciprocal of the relaxation time and shear viscosity (and other transport coefficients) have been derived by Mueller[1] for a gas consisting of loaded sphere molecules. These results express $\frac{1}{\tau}$ and $\frac{1}{\eta}$ as power series in $\frac{\delta}{\sigma}$ where σ is the diameter of the molecule and δ is the displacement of the center of mass from the geometric center. Expressions for the coefficients of the zeroth, first and second order terms have been derived. The results are:

$$\frac{1}{\tau} = 8\pi n\sqrt{\frac{kT}{\pi m}}\left(\frac{\delta}{\sigma}\right)^2\left[\sum_{j^a}(2j^a+1)\,e^{-\epsilon_{j^a}}\right]^{-2}$$

$$\sum_{j^a j^b j'^a j'^b}(\Delta\epsilon)^2(2j^a+1)(2j^b+1)\,e^{-\epsilon_{j^a}-\epsilon_{j^b}}\int \bar{\gamma}^3 e^{-\bar{\gamma}^2} I_{inel}(j^a j^b j'^a j'^b|0)_2\, d\bar{\gamma} \tag{1-1}$$

and

$$\frac{1}{\eta} = \frac{8}{5\sqrt{\pi m k T}}\left[\sum_{j^a}(2j^a+1)\,e^{-\epsilon_{j^a}}\right]^{-2}\sum_{j^a j^b j'^a j'^b}(2j^a+1)(2j^b+1)\,e^{-\epsilon_{j^a}-\epsilon_{j^b}}$$

$$\int\left[\bar{\gamma}^4 g^{(2)}(j^a j^b j'^a j'^b) + \frac{4\pi}{15}(\Delta\epsilon)^2\left(\frac{\delta}{\sigma}\right)^2 I_{inel}(j^a j^b j'^a j'^b|2)_2\right]\bar{\gamma}^3 e^{-\bar{\gamma}^2}\, d\bar{\gamma} \tag{1-2}$$

where

$$I_{inel}(\bar{\ell}^a \bar{\ell}^b \ell^a \ell^b|0)_2 = \frac{1}{3}\frac{\mathcal{H}}{\bar{\mathcal{H}}}\sigma^2\left\{\left(S^{1\bar{\ell}^a}_{\ell^a 00}\right)^2\delta_{\ell^b\bar{\ell}^b} + \left(S^{1\bar{\ell}^b}_{\ell^b 00}\right)^2\delta_{\ell^a\bar{\ell}^a}\right\}$$

$$\sum_\lambda (\lambda+1)\left(\bar{\eta}'_\lambda \eta'_{\lambda+1} + \eta'_\lambda \bar{\eta}'_{\lambda+1}\right) \qquad (1\text{-}3)$$

$$I_{inel}(\bar{\ell}^a \bar{\ell}^b \ell^a \ell^b|2)_2 = \frac{5}{3}\frac{\mathcal{H}}{\bar{\mathcal{H}}}\sigma^2\left\{\left(S^{1\bar{\ell}^a}_{\ell^a 00}\right)^2\delta_{\ell^b\bar{\ell}^b} + \left(S^{1\bar{\ell}^b}_{\ell^b 00}\right)^2\delta_{\ell^a\bar{\ell}^a}\right\}$$

$$\sum_\lambda\left\{\frac{3\lambda(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)}\left[\sqrt{\eta'_{\lambda-1}\bar{\eta}'_\lambda\eta'_{\lambda+1}\bar{\eta}'_{\lambda+2}}\cos(\bar{\eta}_{\lambda+2}-\bar{\eta}_\lambda+\eta_{\lambda+1}-\eta_{\lambda-1})\right.\right.$$

$$\left. + \sqrt{\bar{\eta}'_{\lambda-1}\eta'_\lambda\bar{\eta}'_{\lambda+1}\eta'_{\lambda+2}}\cos(\eta_{\lambda+2}-\eta_\lambda+\bar{\eta}_{\lambda+1}-\bar{\eta}_{\lambda-1})\right]$$

$$-\frac{6(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)(2\lambda+5)}\left[\sqrt{\bar{\eta}'_\lambda\eta'_{\lambda+1}\eta^0_{\lambda+1}\bar{\eta}^0_{\lambda+2}}\cos(\bar{\eta}_{\lambda+2}-\bar{\eta}_\lambda)\right.$$

$$\left. + \sqrt{\eta'_\lambda\bar{\eta}'_{\lambda+1}\bar{\eta}'_{\lambda+1}\eta^0_{\lambda+2}}\cos(\eta_{\lambda+2}-\eta_\lambda)\right]$$

$$\left. + \frac{\lambda(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)}\left[\bar{\eta}'_\lambda\eta'_{\lambda+1}+\eta'_\lambda\bar{\eta}'_{\lambda+1}\right]\right\} \qquad (1\text{-}4)$$

$$g^{(2)}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b) = Q^{(2)}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b)_0 + \left(\frac{\delta}{\sigma}\right)^2\left[Q^{(2)}_{d,A+B+C_1}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b)_2\right.$$

$$\left. + g^{(2)}_{d,C_2}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b)_2 + Q^{(2)}_{inel}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b)_2\right] + \cdots\cdots$$

$$(1\text{-}5)$$

$$Q^{(2)}(\bar{l}^a\bar{l}^b l^a l^b)_0 = \delta_{l^a\bar{l}^a}\,\delta_{l^b\bar{l}^b}\,\frac{4\pi}{\mathcal{H}^2}\sum_{\lambda}\frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)}\sin^2(\eta_{\lambda+2}-\eta_{\lambda}) \tag{1-6}$$

$$Q^{(2)}_{el,A+B+C_1}(\bar{l}^a\bar{l}^b l^a l^b)_2 = -\frac{8\pi}{3}\,\frac{\sigma}{\mathcal{H}}\,\delta_{l^a\bar{l}^a}\,\delta_{l^b\bar{l}^b}\sum_{\lambda}\frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)}$$

$$(\eta'_{\lambda+2}-\eta'_{\lambda})\sin 2(\eta_{\lambda+2}-\eta_{\lambda}) \tag{1-7}$$

$$q^{(2)}_{el,C_2}(\bar{l}^a\bar{l}^b l^a l^b)_2 = \frac{2\pi}{3}\,\frac{\mathcal{H}}{\mathcal{H}}\,\sigma^2\left\{\left(S^{1\bar{l}^a}_{l^a00}\right)^2\delta_{l^b\bar{l}^b} + \left(S^{1\bar{l}^b}_{l^b00}\right)^2\delta_{l^a\bar{l}^a}\right\}$$

$$\sum_{\lambda}\left\{\frac{(\lambda-1)\lambda^2}{(2\lambda-1)(2\lambda+1)}\,\bar{\eta}'_{\lambda}\left[\frac{\eta''_{\lambda-1}}{\eta'_{\lambda-1}}\sin 2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda-2}) - 4\,\eta'_{\lambda-1}\sin^2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda-2})\right]\right.$$

$$+\frac{\lambda(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)}\,\bar{\eta}'_{\lambda}\left[\frac{\eta''_{\lambda-1}}{\eta'_{\lambda-1}}\sin 2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda+2}) - 4\,\eta'_{\lambda-1}\sin^2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda+2})\right]$$

$$+\frac{(\lambda-1)\lambda(\lambda+1)}{(2\lambda-1)(2\lambda+1)}\,\bar{\eta}'_{\lambda}\left[\frac{\eta''_{\lambda+1}}{\eta'_{\lambda+1}}\sin 2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda-2}) - 4\,\eta'_{\lambda+1}\sin^2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda-2})\right]$$

$$\left.+\frac{(\lambda+1)^2(\lambda+2)}{(2\lambda+1)(2\lambda+3)}\,\bar{\eta}'_{\lambda}\left[\frac{\eta''_{\lambda+1}}{\eta'_{\lambda+1}}\sin 2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda+2}) - 4\,\eta'_{\lambda+1}\sin^2(\bar{\eta}_{\lambda}-\bar{\eta}_{\lambda+2})\right]\right\} \tag{1-8}$$

and

$$Q^{(2)}_{inel}(\bar{\ell}^a\bar{\ell}^b\ell^a\ell^b)_2 = \frac{8\pi}{3}\,\frac{\mathcal{H}}{\bar{\mathcal{H}}}\,\sigma^2\left\{\left(S^{1\bar{\ell}^a}_{\ell^a 00}\right)^2\delta_{\ell^b\bar{\ell}^b} + \left(S^{1\bar{\ell}^b}_{\ell^b 00}\right)^2\delta_{\ell^a\bar{\ell}^a}\right\}$$

$$\sum_\lambda\left\{\frac{(\lambda+1)^3}{(2\lambda+1)(2\lambda+3)}\left(\bar{\eta}'_\lambda\eta'_{\lambda+1} + \eta'_\lambda\bar{\eta}'_{\lambda+1}\right) + \frac{2(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)(2\lambda+5)}\right.$$

$$\left[\sqrt{\bar{\eta}'_\lambda\eta'_{\lambda+1}\eta'_{\lambda+1}\bar{\eta}'_{\lambda+2}}\,\cos(\bar{\eta}_{\lambda+2}-\bar{\eta}_\lambda) + \sqrt{\eta'_\lambda\bar{\eta}'_{\lambda+1}\bar{\eta}'_{\lambda+1}\eta'_{\lambda+2}}\,\cos(\eta_{\lambda+2}-\eta_\lambda)\right]$$

$$-\frac{\lambda(\lambda+1)(\lambda+2)}{(2\lambda+1)(2\lambda+3)}\left[\sqrt{\eta'_{\lambda-1}\bar{\eta}'_\lambda\eta'_{\lambda+1}\bar{\eta}'_{\lambda+2}}\,\cos(\bar{\eta}_{\lambda+2}-\bar{\eta}_\lambda+\eta_{\lambda+1}-\eta_{\lambda-1})\right.$$

$$\left.\left.+\sqrt{\bar{\eta}'_{\lambda-1}\eta'_\lambda\bar{\eta}'_{\lambda+1}\eta'_{\lambda+2}}\,\cos(\eta_{\lambda+2}-\eta_\lambda+\bar{\eta}_{\lambda+1}-\bar{\eta}_{\lambda-1})\right]\right\}$$

(1-9)

The series development is not explicit in Eq. 1-2, (see Eq. 1-5) but is explicit in Section IV.

The phase shift is given by

$$\eta_\lambda(\mathcal{H}\sigma) = \tan^{-1}\left[(-1)^{\lambda+1}\frac{J_{\lambda+\frac{1}{2}}(\mathcal{H}\sigma)}{J_{-\lambda-\frac{1}{2}}(\mathcal{H}\sigma)}\right] \qquad (1\text{-}10)$$

where $\mathcal{H}$ is the propogation vector.

Mueller considered the classical limit of Eqs. 1-1 and 1-2. The purpose of this report is to evaluate the strong quantum limit and develop a series valid in the strong quantum region. Specifically we consider an expansion of Eqs. 1-1 and 1-2 in power series in $\frac{1}{\lambda^2}$ where $\lambda = \frac{\hbar}{\sigma\sqrt{2\mu kT}}$.

The first step in considering Eqs. 1-1 and 1-2 is the evaluation of the sums over ℓ^a and ℓ^b. These summations are carried out in section II. The next step is the reduction of Eqs. 1-3 through 1-9 to quantities involving Bessel functions. This is accomplished by expressing the trigonometric functions of η_λ and the derivatives of η_λ entirely in terms of $\tan \eta_\lambda$ and using Eq. 1-10 for $\tan \eta_\lambda$ Once this is accomplished we use the explicit expressions for the low order Bessel functions. This process is carried out in Section III. In the last section we sum the appropriate moments of the cross section over λ . This series in λ is truncated after the first or second term and the terms expanded in a power series in $\mu\sigma$. The integral is then integrated termwise to give a power series in $\frac{1}{\lambda^2}$. The result is $\frac{1}{\gamma}$ and $\frac{1}{\eta}$ as power series in $\frac{1}{\lambda^2}$.

SECTION II

THE SUM OVER $\bar{l}^{a}$ AND $\bar{l}^{b}$

In this section the sum over $\bar{l}^{a}$ and $\bar{l}^{b}$ is carried out for both $\frac{1}{\tau}$ and $\frac{1}{\eta}$. Before evaluating these sums several quantities are defined. These are

$$h = \frac{\hbar}{2\mu} \tag{2-1}$$

where $\hbar$ is Planck's constant and μ is the reduced mass of a pair of colliding molecules a and b, the kinetic energy of relative motion

$$E = h^{2}\mathcal{H}^{2} \tag{2-2}$$

the dimensionless quantities

$$\gamma = \sqrt{\frac{E}{kT}} \tag{2-3}$$

$$\epsilon_{l} = \frac{\Theta}{T} l(l+1) \tag{2-4}$$

where

$$\Theta = \frac{\hbar^{2}}{2Ik} \tag{2-5}$$

and

$$\chi = \epsilon_{l^a} + \epsilon_{l^b} - \epsilon_{\bar{l}^a} - \epsilon_{\bar{l}^b} \tag{2-6}$$

The bar above a quantum number or variable indicates the value before collision. The superscript a or b serves to distinguish between the two colliding molecules.

Conservation of energy requires that

$$\gamma^2 = \bar{\gamma}^2 - \chi \tag{2-7}$$

In the development that follows it is more convenient to work with the variable

$$z = \mathcal{H}\sigma \tag{2-8}$$

rather than γ . We combine this definition with Eqs. 2-2 and 2-3 to obtain γ in terms of z

$$\gamma = \frac{h}{\sigma\sqrt{kT}}\, z$$

We then define

$$\lambda = \frac{\hbar}{\sigma\sqrt{2\mu kT}} \tag{2-9}$$

so that

$$\gamma = \lambda z \qquad (2\text{-}10)$$

We now proceed to evaluate the sum over l^a and l^b for $\frac{1}{\tau}$. Substitution of Eq. 1-3 into 1-1 gives the following:

$$\frac{1}{\tau} = \frac{8\pi n\sigma^2}{3}\sqrt{\frac{kT}{\pi m}}\left(\frac{\delta}{\sigma}\right)^2\left[\sum_{\bar{l}^a}(2\bar{l}^a+1)\,e^{-\epsilon_{\bar{l}^a}}\right]^{-2}$$

$$\sum_{\bar{l}^a\bar{l}^b l^a l^b}(2\bar{l}^a+1)(2\bar{l}^b+1)\,e^{-\epsilon_{\bar{l}^a}-\epsilon_{\bar{l}^b}}\,x^2\left\{\left(S^{1\bar{l}^a}_{l^a 00}\right)^2\delta_{l^b\bar{l}^b}+\left(S^{1\bar{l}^b}_{l^b 00}\right)^2\delta_{l^a\bar{l}^a}\right\}$$

$$\int\gamma\bar{\gamma}^2\sum_{\lambda}(\lambda+1)\left(\eta'_\lambda\bar{\eta}'_{\lambda+1}+\bar{\eta}'_\lambda\eta'_{\lambda+1}\right)e^{-\bar{\gamma}^2}\,d\bar{\gamma} \qquad (2\text{-}11)$$

where $\frac{\gamma}{\bar{\gamma}}$ has been substituted for $\frac{\mu}{\bar{\mu}}$.

Consider the quantity

$$x^2\left\{\left(S^{1\bar{l}^a}_{l^a 00}\right)^2\delta_{l^b\bar{l}^b}+\left(S^{1\bar{l}^b}_{l^b 00}\right)^2\delta_{l^a\bar{l}^a}\right\}\int\gamma\bar{\gamma}^2 e^{-\bar{\gamma}^2}$$

$$\sum_{\lambda}(\lambda+1)\left(\eta'_\lambda\bar{\eta}'_{\lambda+1}+\bar{\eta}'_\lambda\eta'_{\lambda+1}\right)d\bar{\gamma} \qquad (2\text{-}12)$$

which is the factor in the terms of the sum which depends upon l^a and l^b. When the sum over l^a and l^b is carried out we obtain only four non-zero terms because the bracketed quantity is zero in all but the four cases listed in Table 2-1.

TABLE 2-1

$\Delta\ell^a = \ell^a - \bar{\ell}^a$	$\Delta\ell^b = \ell^b - \bar{\ell}^b$	$\left\{\left(S^{1\bar{\ell}^a}_{\ell^a 00}\right)^2 \delta_{\ell^b \bar{\ell}^b} + \left(S^{1\bar{\ell}^b}_{\ell^b 00}\right)^2 \delta_{\ell^a \bar{\ell}^a}\right\}$	χ
+1	0	$\frac{\bar{\ell}^a+1}{2\bar{\ell}^a+1}$	$\frac{2\Theta}{T}(\bar{\ell}^a+1)$
-1	0	$\frac{\bar{\ell}^a}{2\bar{\ell}^a+1}$	$-\frac{2\Theta}{T}\bar{\ell}^a$
0	+1	$\frac{\bar{\ell}^b+1}{2\bar{\ell}^b+1}$	$\frac{2\Theta}{T}(\bar{\ell}^b+1)$
0	-1	$\frac{\bar{\ell}^b}{2\bar{\ell}^b+1}$	$-\frac{2\Theta}{T}\bar{\ell}^b$

It is important to note the dependence of the integral in Eq. 2-12 on the four quantum numbers $\bar{\ell}^a \bar{\ell}^b \ell^a \ell^b$. The η_λ and $\bar{\eta}_\lambda$ can all be written in terms of γ and $\bar{\gamma}$, and γ can be written $\bar{\gamma} - \chi$ (Eq. 2-7). Thus, the only dependence of the integral on the ℓ quantum numbers is through χ . The values of χ in the four non-zero cases are listed in Table 2-1. We therefore define

$$s\left(\chi \frac{T}{2\Theta}\right) = \chi^2 \int \bar{\gamma}^2 \gamma e^{-\bar{\gamma}^2} \sum_\lambda (\lambda+1)\left(\eta'_\lambda \bar{\eta}'_{\lambda+1} + \bar{\eta}'_\lambda \eta'_{\lambda+1}\right) d\bar{\gamma} \qquad (2\text{-}13)$$

and

$$\rho_\ell = \frac{-\ell\, \rho(-\ell) + (\ell+1)\, \rho(\ell+1)}{2\ell+1} \tag{2-14}$$

Thus, after summing over $\bar{\ell}^a$ and $\bar{\ell}^b$, the expression 2-12 becomes simply

$$\rho_{\bar{\ell}^a} + \rho_{\bar{\ell}^b} \tag{2-15}$$

and Eq. 2-11 becomes

$$\frac{1}{\tau} = \frac{8\pi n \sigma^2}{3} \sqrt{\frac{kT}{\pi m}} \left(\frac{S}{\sigma}\right)^2 \frac{\sum_{\bar{\ell}^a \bar{\ell}^b} (2\bar{\ell}^a+1)(2\bar{\ell}^b+1)\, e^{-\varepsilon_{\bar{\ell}^a}-\varepsilon_{\bar{\ell}^b}} \left(\rho_{\bar{\ell}^a}+\rho_{\bar{\ell}^b}\right)}{\left[\sum_{\bar{\ell}^a} (2\bar{\ell}^a+1)\, e^{-\varepsilon_{\bar{\ell}^a}}\right]^2} \tag{2-16}$$

But

$$\sum_{\bar{\ell}^a}\sum_{\bar{\ell}^b} (2\bar{\ell}^a+1)(2\bar{\ell}^b+1)\, e^{-\varepsilon_{\bar{\ell}^a}-\varepsilon_{\bar{\ell}^b}} \left(\rho_{\bar{\ell}^a}+\rho_{\bar{\ell}^b}\right) = 2 \sum_{\bar{\ell}^b} (2\bar{\ell}^b+1)\, e^{-\varepsilon_{\bar{\ell}^b}} \rho_{\bar{\ell}^b} \sum_{\bar{\ell}^a} (2\bar{\ell}^a+1)\, e^{-\varepsilon_{\bar{\ell}^a}} \tag{2-17}$$

Therefore

$$\frac{1}{\tau} = \frac{8\pi n \sigma^2}{3} \sqrt{\frac{kT}{\pi m}} \left(\frac{S}{\sigma}\right)^2 \frac{2 \sum_{\ell} (2\ell+1)\, e^{-\varepsilon_\ell} \rho_\ell}{\sum_{\ell} (2\ell+1)\, e^{-\varepsilon_\ell}} \tag{2-18}$$

Explicit expressions for the $\mathcal{A}_\ell$ are obtained in Section IV.

We now turn to the evaluation of $\frac{1}{\eta}$. Substitution of Eq. 1-5 into Eq. 1-2 gives

$$\frac{1}{\eta} = \frac{8}{5\sqrt{\pi m K T}}\left[\sum_{J^a}(2J^a+1)\,e^{-\epsilon_{J^a}}\right]^{-2}\sum_{J^a J^b \ell^a \ell^b}(2J^a+1)(2J^b+1)\,e^{-\epsilon_{J^a}-\epsilon_{J^b}}$$

$$\int\Big\{\Big[\bar{\gamma}^4 Q^{(2)}(J^aJ^b\ell^a\ell^b)_0 + \left(\frac{\delta}{\sigma}\right)^2\Big[\bar{\gamma}^4 Q^{(2)}_{\mathcal{A}, A+B+C_1}(J^aJ^b\ell^a\ell^b)_2 + \bar{\gamma}^4 \mathcal{G}^{(2)}_{\mathcal{A}, C_2}(J^aJ^b\ell^a\ell^b)_2$$

$$+\bar{\gamma}^4 Q^{(2)}_{inel}(J^aJ^b\ell^a\ell^b)_2 + \frac{4\pi}{15}x^2 I_{inel}(J^aJ^b\ell^a\ell^b|2)_2\Big]+\cdots\Big\}\,\bar{\gamma}^3 e^{-\bar{\gamma}^2}\,d\bar{\gamma}$$

(2-19)

Grouping this according to powers of $\frac{\delta}{\sigma}$ gives

$$\frac{1}{\eta} = \frac{8}{5\sqrt{\pi m K T}}\left[\sum_{J^a}(2J^a+1)e^{-\epsilon_{J^a}}\right]^{-2}\sum_{J^a J^b \ell^a \ell^b}(2J^a+1)(2J^b+1)\,e^{-\epsilon_{J^a}-\epsilon_{J^b}}$$

$$\int \bar{\gamma}^7 Q^{(2)}(J^aJ^b\ell^a\ell^b)_0\, e^{-\bar{\gamma}^2}\,d\bar{\gamma} + \frac{8}{5\sqrt{\pi m K T}}\left(\frac{\delta}{\sigma}\right)^2\left[\sum_{J^a}(2J^a+1)\,e^{-\epsilon_{J^a}}\right]^{-2}$$

$$\sum_{J^a J^b \ell^a \ell^b}(2J^a+1)(2J^b+1)\,e^{-\epsilon_{J^a}-\epsilon_{J^b}}\int\Big[\bar{\gamma}^7 Q^{(2)}_{\mathcal{A}, A+B+C_1}(J^aJ^b\ell^a\ell^b)_2 + \bar{\gamma}^7 \mathcal{G}^{(2)}_{\mathcal{A}, C_2}(J^aJ^b\ell^a\ell^b)_2$$

$$+\bar{\gamma}^7 Q^{(2)}_{inel}(J^aJ^b\ell^a\ell^b)_2 + \bar{\gamma}^3\frac{4\pi}{15}x^2 I_{inel}(J^aJ^b\ell^a\ell^b|2)_2\Big]\,e^{-\bar{\gamma}^2}\,d\bar{\gamma} + \cdots\cdots$$

(2-20)

which can be written as

$$\frac{1}{\eta} = \frac{1}{\eta_0} + \left(\frac{\delta}{\sigma}\right)^2 \frac{1}{\eta_2} + \cdots \tag{2-21}$$

This is an explicit formulation of $\frac{1}{\eta}$ as a power series in $\frac{\delta}{\sigma}$. We first consider the zeroth order term. This is the quantum expression for the rigid sphere case, which was first obtained by Uehling[2]. From Eqs.(2-21)and(2-20), it is seen that

$$\frac{1}{\eta_0} = \frac{8}{5\sqrt{\pi m k T}} \left[\sum_{\bar{J}^a} (2\bar{J}^a+1) e^{-\epsilon_{\bar{J}^a}}\right]^{-2} \sum_{\bar{J}^a \bar{J}^b J^a J^b} (2\bar{J}^a+1)(2\bar{J}^b+1) e^{-\epsilon_{\bar{J}^a} - \epsilon_{\bar{J}^b}}$$

$$\int \bar{\gamma}^7 Q^{(2)}(\bar{J}^a\bar{J}^b J^a J^b)_0 \, e^{-\bar{\gamma}^2} d\bar{\gamma} \tag{2-22}$$

Substitution of Eq. 1-6 into Eq. 2-22 gives

$$\frac{1}{\eta_0} = \frac{8}{5\sqrt{\pi m k T}} \left[\sum_{\bar{J}^a} (2\bar{J}^a+1) e^{-\epsilon_{\bar{J}^a}}\right]^{-2} \sum_{\bar{J}^a \bar{J}^b J^a J^b} (2\bar{J}^a+1)(2\bar{J}^b+1) e^{-\epsilon_{\bar{J}^a} - \epsilon_{\bar{J}^b}}$$

$$\delta_{J^a \bar{J}^a} \delta_{J^b \bar{J}^b} (4\pi) \int \frac{\bar{\gamma}^7}{\varkappa^2} \sum_{\lambda} \frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)} \sin^2(\eta_{\lambda+2} - \eta_\lambda) \, e^{-\bar{\gamma}^2} d\bar{\gamma} \tag{2-23}$$

The factor $\delta_{J^a\bar{J}^a}\delta_{J^b\bar{J}^b}$ requires the internal quantum number for each molecule to be the same before and after the collision. Thus $\gamma = \bar{\gamma}$ and $\chi = 0$. Therefore the integrand is independent of $\bar{J}^a$ and $\bar{J}^b$ for the only term in the sum over J^a and J^b which is not zero. Hence, we can factor the integral out of the sum over $\bar{J}^a$ and $\bar{J}^b$. Then, after interchanging the order of the sum over λ and the integration, we obtain

$$\frac{1}{\eta_0} = \frac{8}{5\sqrt{\pi m k T}}\left[\sum_{j^a}(2j^a+1)\,e^{-\epsilon_{j^a}}\right]^{-2}(4\pi)\left[\sum_{\lambda}\frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)}\right.$$

$$\left.\int\frac{\gamma^7}{\mathcal{H}^2}\sin^2(\eta_{\lambda+2}-\eta_\lambda)\,e^{-\gamma^2}d\gamma\right]\sum_{j^a j^b}(2j^a+1)(2j^b+1)\,e^{-\epsilon_{j^a}-\epsilon_{j^b}} \tag{2-24}$$

But

$$\sum_{j^a j^b}(2j^a+1)(2j^b+1)e^{-\epsilon_{j^a}-\epsilon_{j^b}} = \left[\sum_{j^a}(2j^a+1)\,e^{-\epsilon_{j^a}}\right]^2 \tag{2-25}$$

Thus

$$\frac{1}{\eta_0} = \frac{32\pi}{5\sqrt{\pi m k T}}\sum_{\lambda}\frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)}\int\frac{\gamma^7}{\mathcal{H}^2}\sin^2(\eta_{\lambda+2}-\eta_\lambda)\,e^{-\gamma^2}d\gamma \tag{2-26}$$

This result is considered further in section IV.

We now consider the second order term in Eq. 2-21

$$\frac{1}{\eta_2} = \frac{8}{5\sqrt{\pi m k T}}\left[\sum_{j^a}(2j^a+1)e^{-\epsilon_{j^a}}\right]^{-2}\sum_{j^a j^b j'^a j'^b}(2j^a+1)(2j^b+1)\,e^{-\epsilon_{j^a}-\epsilon_{j^b}}$$

$$\int\Big[\bar\gamma^7 Q^{(2)}_{el,A+B+C_1}(j^a j^b j'^a j'^b)_2 + \bar\gamma^7 Q^{(2)}_{el,C_2}(j^a j^b j'^a j'^b)_2 + \bar\gamma^7 Q^{(2)}_{ind}(j^a j^b j'^a j'^b)_2$$

$$\left.\frac{4\pi}{15}x^2\bar\gamma^3 I_{ind}(j^a j^b j'^a j'^b|2)_2\right]e^{-\bar\gamma^2}d\bar\gamma \tag{2-27}$$

We rewrite this as

$$\frac{1}{\eta_2} = \frac{8}{5\sqrt{\pi m k T}}\left[\sum_{\bar{j}^a}(2\bar{j}^a+1)e^{-\epsilon_{\bar{j}^a}}\right]^{-2}\sum_{\bar{j}^a\bar{j}^b j^a j^b}(2\bar{j}^a+1)(2\bar{j}^b+1)e^{-\epsilon_{\bar{j}^a}-\epsilon_{\bar{j}^b}}$$

$$\int \bar{\gamma}^7 Q^{(2)}_{el,A+B+C_1}(\bar{j}^a\bar{j}^b j^a j^b)_2\, e^{-\bar{\gamma}^2}d\bar{\gamma} + \frac{8}{5\sqrt{\pi m k T}}\left[\sum_{\bar{j}^a}(2\bar{j}^a+1)e^{-\epsilon_{\bar{j}^a}}\right]^{-2}\sum_{\bar{j}^a\bar{j}^b j^a j^b}$$

$$(2\bar{j}^a+1)(2\bar{j}^b+1)e^{-\epsilon_{\bar{j}^a}-\epsilon_{\bar{j}^b}}\int\Big[\bar{\gamma}^7 Q^{(2)}_{el,C_2}(\bar{j}^a\bar{j}^b j^a j^b)_2$$

$$+\ \bar{\gamma}^7 Q^{(2)}_{ind}(\bar{j}^a\bar{j}^b j^a j^b)_2 + \frac{4T}{16}X^2\bar{\gamma}^3 I_{ind}(\bar{j}^a\bar{j}^b j^a j^b|2)_2\Big]e^{-\bar{\gamma}^2}d\bar{\gamma} \tag{2-28}$$

When we sum the first term over $\bar{j}^a$ and $\bar{j}^b$ we again obtain only one non-zero term, because of the factor, $\delta_{j^a\bar{j}^a}\delta_{j^b\bar{j}^b}$, in Eq. 1-7, the term for which $j^a=\bar{j}^a$ and $j^b=\bar{j}^b$. Since $X=0$, the integral does not depend upon either $\bar{j}^a$ or $\bar{j}^b$ and can be factored out of the sum. The first term in $\frac{1}{\eta_2}$ then becomes

$$\frac{8}{5\sqrt{\pi m k T}}\int\bar{\gamma}^7 Q^{(2)}_{el,A+B+C_1}(\bar{j}^a\bar{j}^b j^a j^b)_2\, e^{-\bar{\gamma}^2}d\bar{\gamma}\left[\sum_{\bar{j}^a}(2\bar{j}^a+1)e^{-\epsilon_{\bar{j}^a}}\right]^{-2}\sum_{\bar{j}^a\bar{j}^b}(2\bar{j}^a+1)(2\bar{j}^b+1)e^{-\epsilon_{\bar{j}^a}-\epsilon_{\bar{j}^b}}$$

As in the $\frac{1}{\eta_0}$ term the last two sums cancel leaving

$$\frac{8}{5\sqrt{\pi m k T}}\int\bar{\gamma}^7 Q^{(2)}_{el,A+B+C_1}(\bar{j}^a\bar{j}^b j^a j^b)_2\, e^{-\bar{\gamma}^2}d\bar{\gamma} \tag{2-29}$$

When we sum over j^a and j^b in the second term of Eq. 2-28 we obtain only four terms because the first bracketed quantity in Eqs. 1-8, 1-9, and 1-4 is zero except in the four cases listed in Table 2-1. The development from this point is analogous to the development of $\frac{1}{\tau}$. For simplicity we factor the bracketed term out of Eqs. 1-8, 1-9, and 1-4 so that the second term in Eq. 2-28 becomes

$$\frac{8}{5\sqrt{\pi m k T}}\left[\sum_{j^a}(2j^a+1)e^{-\epsilon_{j^a}}\right]^{-2}\sum_{j^a j^b l^a l^b}(2l^a+1)(2l^b+1)\,e^{-\epsilon_{j^a}-\epsilon_{j^b}}$$

$$\left\{\left(S^{1j^a}_{l^a 00}\right)^2\delta_{l^b j^b}+\left(S^{1j^b}_{l^b 00}\right)^2\delta_{l^a j^a}\right\}\int\Big[\bar{\gamma}^7\,\tilde{\mathcal{S}}^{(2)}_{el,C_2}(j^a j^b l^a l^b)_2$$

$$+\bar{\gamma}^7\,\tilde{Q}^{(2)}_{inel}(j^a j^b l^a l^b)_2+\bar{\gamma}^3\frac{4\pi}{15}x^2\,\tilde{I}_{inel}(j^a j^b l^a l^b|2)_2\Big]e^{-\bar{\gamma}^2}d\bar{\gamma} \tag{2-30}$$

where the tilda above a quantity indicates that the bracketed term has been factored out and removed. The factor in the terms of the sum which depends upon l^a and l^b is then

$$\left\{\left(S^{1j^a}_{l^a 00}\right)^2\delta_{l^b j^b}+\left(S^{1j^b}_{l^b 00}\right)^2\delta_{l^a j^a}\right\}\int\Big[\bar{\gamma}^7\,\tilde{\mathcal{S}}^{(2)}_{el,C_2}(j^a j^b l^a l^b)_2$$

$$+\bar{\gamma}^7\,\tilde{Q}^{(2)}_{inel}(j^a j^b l^a l^b)_2+\bar{\gamma}^3\frac{4\pi}{15}x^2\,\tilde{I}_{inel}(j^a j^b l^a l^b|2)_2\Big]e^{-\bar{\gamma}^2}d\bar{\gamma} \tag{2-31}$$

We now define a quantity Γ analogous to Δ

$$\Gamma\left(x\frac{T}{2\Theta}\right)=\int\Big[\bar{\gamma}^7\,\tilde{\mathcal{S}}^{(2)}_{el,C_2}(j^a j^b l^a l^b)_2+\bar{\gamma}^7\,\tilde{Q}^{(2)}_{inel}(j^a j^b l^a l^b)_2$$

$$+\bar{\gamma}^3\frac{4\pi}{15}x^2\,\tilde{I}_{inel}(j^a j^b l^a l^b|2)_2\Big]e^{-\bar{\gamma}^2}d\bar{\gamma} \tag{2-32}$$

and also define

$$r_\ell = \frac{-\ell\, r(-\ell) + (\ell+1)\, r(\ell+1)}{2\ell+1} \tag{2-33}$$

After summing over ℓ^a and ℓ^b the expression (2-31) then becomes

$$r_{j^a} + r_{j^b} \tag{2-34}$$

and the second term in $\frac{1}{\eta_2}$ becomes

$$\frac{8}{5\sqrt{\pi m K T}} \frac{\sum_{j^a j^b} (2j^a+1)(2j^b+1)\, e^{-\epsilon_{j^a} - \epsilon_{j^b}} (r_{j^a} + r_{j^b})}{\left[\sum_{j^a} (2j^a+1)\, e^{-\epsilon_{j^a}}\right]^2} \tag{2-35}$$

As in the analogous expression for $\frac{1}{\gamma}$, Eq. (2-16), the sum reduces and we obtain

$$\frac{8}{5\sqrt{\pi m K T}} \frac{2\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell}\, r_\ell}{\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell}} \tag{2-36}$$

This result, together with Eq. (2-29) gives

$$\frac{1}{\eta_2} = \frac{8}{5\sqrt{\pi m K T}} \int \bar{\gamma}^7\, \tilde{Q}^{(2)}_{\ell_1 A+B+C_1} (j^a j^b j'^a j'^b)_2\, e^{-\bar{\gamma}^2}\, d\bar{\gamma}$$

$$+ \frac{8}{5\sqrt{\pi m K T}} \frac{2\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell}\, r_\ell}{\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell}} \tag{2-37}$$

Explicit expressions for the r_ℓ are obtained in section IV.

SECTION III

EVALUATION OF THE CROSS SECTION AND ITS MOMENTS

The next step in the evolution of the transport coefficients is the reduction of Eqs. 1-3 through 1-9 to expressions involving only polynomials in z . This is accomplished by expressing the trigonometric functions of η_λ and the derivatives of η_λ which occur in Eqs. (1-3) through (1-9) in terms of Bessel functions. The Bessel functions are then reduced to polynomials. For example,

$$\sin 2(\eta_{\lambda+2}-\eta_\lambda) = 2\,\frac{(1+\tan\eta_{\lambda+2}\tan\eta_\lambda)(\tan\eta_{\lambda+2}-\tan\eta_\lambda)}{(1+\tan^2\eta_{\lambda+2})(1+\tan^2\eta_\lambda)} \qquad (3\text{-}1)$$

With the aid of Eq. 1-10 we then find that

$$\sin 2(\eta_{\lambda+2}-\eta_\lambda) = 2\left[1+(-1)^{\lambda+1}\frac{J_{(\lambda+2)+\frac{1}{2}}}{J_{-(\lambda+2)-\frac{1}{2}}}(-1)^{\lambda+1}\frac{J_{\lambda+\frac{1}{2}}}{J_{-\lambda-\frac{1}{2}}}\right]$$

$$\cdot\,\frac{\left[(-1)^{\lambda+1}\dfrac{J_{(\lambda+2)+\frac{1}{2}}}{J_{-(\lambda+2)-\frac{1}{2}}}-(-1)^{\lambda+1}\dfrac{J_{\lambda+\frac{1}{2}}}{J_{-\lambda-\frac{1}{2}}}\right]}{\left[1+\dfrac{J^2_{(\lambda+2)+\frac{1}{2}}}{J^2_{-(\lambda+2)-\frac{1}{2}}}\right]\left[1+\dfrac{J^2_{\lambda+\frac{1}{2}}}{J^2_{-\lambda-\frac{1}{2}}}\right]}$$

Multiplying through by $J^2_{-(\lambda+2)-\frac{1}{2}}\,J^2_{-\lambda-\frac{1}{2}}$ gives

$$\sin 2(\eta_{\lambda+2}-\eta_\lambda) = \frac{2(-1)^{\lambda+1}\left[J_{-(\lambda+2)-\frac{1}{2}}J_{-\lambda-\frac{1}{2}}+J_{(\lambda+2)+\frac{1}{2}}J_{\lambda+\frac{1}{2}}\right]\left[J_{-\lambda-\frac{1}{2}}J_{(\lambda+2)+\frac{1}{2}}-J_{-(\lambda+2)-\frac{1}{2}}J_{\lambda+\frac{1}{2}}\right]}{\left[J^2_{-(\lambda+2)-\frac{1}{2}}+J^2_{(\lambda+2)+\frac{1}{2}}\right]\left[J^2_{-\lambda-\frac{1}{2}}+J^2_{\lambda+\frac{1}{2}}\right]} \qquad (3\text{-}2)$$

Similar expressions can be derived for the other trigonometric functions of η_λ and derivatives of η_λ.

Before writing these down, however, it is convenient to simplify two terms which appear frequently in these expressions. These terms have the general form

$$J_\nu J_{1-\nu} + J_{-\nu} J_{\nu-1}$$

and

$$J_{-\nu} J_{\nu-2} - J_\nu J_{2-\nu}$$

The first is simply Lommel's formula[3]

$$J_\nu(z) J_{1-\nu}(z) + J_{-\nu}(z) J_{\nu-1}(z) = \left(\frac{2}{\pi z}\right) \sin \nu\pi = (-1)^{\nu-\frac{1}{2}} \left(\frac{2}{\pi z}\right) \tag{3-3}$$

Using this equation and the recurrence relation[4]

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{3-4}$$

it is easy to derive a similar expression for $J_{-\nu} J_{\nu-2} - J_\nu J_{2-\nu}$

$$J_{-\nu}(z) J_{\nu-2}(z) - J_\nu(z) J_{2-\nu}(z) = (-1)^{\nu-\frac{1}{2}} \frac{4(\nu-1)}{\pi z^2} \tag{3-5}$$

Therefore, the second bracketed term in Eq. 3-2 becomes

$$J_{-\lambda-\frac{1}{2}}\, J_{(\lambda+2)+\frac{1}{2}} - J_{\lambda+\frac{1}{2}}\, J_{-(\lambda+2)-\frac{1}{2}} = (-1)^{\lambda+1}\, \frac{4(\lambda+\frac{3}{2})}{\pi z^2}$$

and Eq. 3-2 becomes

$$\sin 2(\eta_{\lambda+2} - \eta_\lambda) = \frac{4(2\lambda+3)}{\pi z^2}\, \frac{J_{-(\lambda+2)-\frac{1}{2}}\, J_{-\lambda-\frac{1}{2}} + J_{(\lambda+2)+\frac{1}{2}}\, J_{\lambda+\frac{1}{2}}}{\left[J^2_{-(\lambda+2)-\frac{1}{2}} + J^2_{(\lambda+2)+\frac{1}{2}}\right]\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]} \qquad (3\text{-}6)$$

Similar derivations yield the following results:

$$\sin^2(\eta_{\lambda+2} - \eta_\lambda) = \frac{4(2\lambda+3)^2}{\pi^2 z^4}\, \frac{1}{\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]\left[J^2_{-(\lambda+2)-\frac{1}{2}} + J^2_{(\lambda+2)+\frac{1}{2}}\right]} \qquad (3\text{-}7)$$

$$\sin^2(\bar{\eta}_\lambda - \bar{\eta}_{\lambda-2}) = \frac{4(2\lambda-1)^2}{\pi^2 \bar{z}^4}\, \frac{1}{\left[\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{J}^2_{\lambda+\frac{1}{2}}\right]\left[\bar{J}^2_{(\lambda-2)-\frac{1}{2}} + \bar{J}^2_{(\lambda-2)+\frac{1}{2}}\right]} \qquad (3\text{-}8)$$

$$\sin^2(\bar{\eta}_\lambda - \bar{\eta}_{\lambda+2}) = \frac{4(2\lambda+3)^2}{\pi^2 \bar{z}^4}\, \frac{1}{\left[\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{J}^2_{\lambda+\frac{1}{2}}\right]\left[\bar{J}^2_{-(\lambda+2)-\frac{1}{2}} + \bar{J}^2_{(\lambda+2)+\frac{1}{2}}\right]} \qquad (3\text{-}9)$$

$$\sin 2(\eta_{\lambda+2} - \eta_\lambda) = \frac{4(2\lambda+3)}{\pi z^2}\, \frac{J_{(\lambda+2)+\frac{1}{2}}\, J_{\lambda+\frac{1}{2}} + J_{-(\lambda+2)-\frac{1}{2}}\, J_{-\lambda-\frac{1}{2}}}{\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]\left[J^2_{-(\lambda+2)-\frac{1}{2}} + J^2_{(\lambda+2)+\frac{1}{2}}\right]} \qquad (3\text{-}10)$$

$$\sin 2(\bar{\eta}_\lambda - \bar{\eta}_{\lambda-2}) = \frac{4(2\lambda-1)}{\pi \bar{z}^2} \; \frac{\bar{\bar{J}}_{\lambda+\frac{1}{2}} \bar{\bar{J}}_{(\lambda-2)+\frac{1}{2}} + \bar{\bar{J}}_{\lambda-\frac{1}{2}} \bar{\bar{J}}_{-(\lambda-2)-\frac{1}{2}}}{\left[\bar{\bar{J}}^2_{-\lambda-\frac{1}{2}} + \bar{\bar{J}}^2_{\lambda+\frac{1}{2}}\right]\left[\bar{\bar{J}}^2_{-(\lambda-2)-\frac{1}{2}} + \bar{\bar{J}}^2_{(\lambda-2)+\frac{1}{2}}\right]} \tag{3-11}$$

$$\sin 2(\bar{\eta}_\lambda - \bar{\eta}_{\lambda+2}) = -\frac{4(2\lambda+3)}{\pi \bar{z}^2} \; \frac{\bar{\bar{J}}_{(\lambda+2)+\frac{1}{2}} \bar{J}_{\lambda+\frac{1}{2}} + \bar{J}_{-(\lambda+2)-\frac{1}{2}} \bar{J}_{-\lambda-\frac{1}{2}}}{\left[\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{J}^2_{\lambda+\frac{1}{2}}\right]\left[\bar{J}^2_{-(\lambda+2)-\frac{1}{2}} + \bar{J}^2_{(\lambda+2)+\frac{1}{2}}\right]} \tag{3-12}$$

$$\cos(\bar{\eta}_{\lambda+2} - \bar{\eta}_\lambda) = \frac{\bar{\bar{J}}_{(\lambda+2)+\frac{1}{2}} \bar{\bar{J}}_{\lambda+\frac{1}{2}} + \bar{J}_{-(\lambda+2)-\frac{1}{2}} \bar{J}_{-\lambda-\frac{1}{2}}}{\left[\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{\bar{J}}^2_{\lambda+\frac{1}{2}}\right]^{1/2}\left[\bar{\bar{J}}^2_{-(\lambda+2)-\frac{1}{2}} + \bar{\bar{J}}^2_{(\lambda+2)+\frac{1}{2}}\right]^{1/2}} \tag{3-13}$$

$$\cos(\bar{\eta}_{\lambda+2} - \bar{\eta}_\lambda + \eta_{\lambda+1} - \eta_{\lambda-1}) = \left\{\left[\bar{\bar{J}}_{(\lambda+2)+\frac{1}{2}} \bar{J}_{\lambda+\frac{1}{2}} + \bar{J}_{-(\lambda+2)-\frac{1}{2}} \bar{J}_{-\lambda-\frac{1}{2}}\right]\right.$$

$$\left.\left[J_{(\lambda+1)+\frac{1}{2}} J_{(\lambda-1)+\frac{1}{2}} + J_{-(\lambda+1)-\frac{1}{2}} J_{-(\lambda-1)-\frac{1}{2}}\right] - \frac{4(2\lambda+3)(2\lambda+1)}{\pi^2 z^2 \bar{z}^2}\right\} \Big/$$

$$\left\{\left(J^2_{-(\lambda-1)-\frac{1}{2}} + J^2_{(\lambda-1)+\frac{1}{2}}\right)\left(\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{J}^2_{\lambda+\frac{1}{2}}\right)\left(J^2_{-(\lambda+1)-\frac{1}{2}} + J^2_{(\lambda+1)+\frac{1}{2}}\right)\left(\bar{J}^2_{-(\lambda+2)-\frac{1}{2}} + \bar{J}^2_{(\lambda+2)+\frac{1}{2}}\right)\right\}^{1/2} \tag{3-14}$$

$$\eta'_\lambda = -\frac{2}{\pi z} \; \frac{1}{\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]} \tag{3-15}$$

$$\frac{\eta''_\lambda}{\eta'_\lambda} = -\frac{1}{z} - \frac{\left[J_{\lambda+\frac{1}{2}} J_{(\lambda-1)+\frac{1}{2}} - J_{-\lambda-\frac{1}{2}} J_{-(\lambda-1)-\frac{1}{2}}\right] - \left[J_{\lambda+\frac{1}{2}} J_{(\lambda+1)+\frac{1}{2}} - J_{-\lambda-\frac{1}{2}} J_{-(\lambda+1)-\frac{1}{2}}\right]}{\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]} \tag{3-16}$$

For small values of λ the remaining combinations of Bessel functions can be reduced to polynomials rather easily using Eq. 3-4 in conjunction with the two equations

$$J_{1/2}(z) = \frac{2}{\pi z} \sin z$$

$$J_{-\frac{1}{2}}(z) = \frac{2}{\pi z} \cos z$$

However, for larger values of λ this process becomes rather unwieldy. A simpler procedure makes use of the following two formulas[5],

$$J_{n+\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \Bigg[\sin\left(z - \tfrac{1}{2} n\pi\right) \sum_{r=0}^{\leq \frac{1}{2} n} \frac{(-1)^r (n+2r)!}{(2r)!\,(n-2r)!\,(2z)^{2r}}$$

$$+ \cos\left(z - \tfrac{1}{2} n\pi\right) \sum_{r=0}^{\leq \frac{1}{2}(n-1)} \frac{(-1)^r (n+2r+1)!}{(2r+1)!\,(n-2r-1)!\,(2z)^{2r+1}} \Bigg]$$

$$J_{-n-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \Bigg[\cos\left(z + \tfrac{1}{2} n\pi\right) \sum_{r=0}^{\leq \frac{1}{2} n} \frac{(-1)^r (n+2r)!}{(2r)!\,(n-2r)!\,(2z)^{2r}}$$

$$- \sin\left(z + \tfrac{1}{2} n\pi\right) \sum_{r=0}^{\leq \frac{1}{2}(n-1)} \frac{(-1)^r (n+2r+1)!}{(2r+1)!\,(n-2r-1)!\,(2z)^{2r+1}} \Bigg]$$

As an example of how these are used, let us consider $J^2_{-n-\frac{1}{2}} + J^2_{n+\frac{1}{2}}$

To facilitate the writing, let

$$A_{nr} = \frac{(-1)^r (n+2r)!}{(2r)!\,(n-2r)!\,(2z)^{2r}}$$

and

$$B_{nr} = \frac{(-1)^r (n+2r+1)!}{(2r+1)!\,(n-2r-1)!\,(2z)^{2r+1}}$$

so

$$J^2_{n+\frac{1}{2}} = \left(\frac{2}{\pi z}\right)\left[\sin(z-\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}n} A_{nr} + \cos(z-\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}(n-1)} B_{nr}\right]^2$$

$$J^2_{-n-\frac{1}{2}} = \left(\frac{2}{\pi z}\right)\left[\cos(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}n} A_{nr} - \sin(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}(n-1)} B_{nr}\right]^2$$

Before adding these two equations we note that

$$\sin(z-\tfrac{1}{2}n\pi) = (-1)^n \sin(z+\tfrac{1}{2}n\pi)$$

and

$$\cos(z-\tfrac{1}{2}n\pi) = (-1)^n \cos(z+\tfrac{1}{2}n\pi)$$

so

$$J^2_{n+\frac{1}{2}} = \left(\frac{2}{\pi z}\right)\left[\sin(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}n} A_{nr} + \cos(z+\tfrac{1}{2}n\pi)\sum_{r=0}^{\leq\frac{1}{2}(n-1)} B_{nr}\right]^2$$

Clearly, then

$$J^2_{-n-\frac{1}{2}} + J^2_{n+\frac{1}{2}} = \left(\frac{2}{\pi z}\right)\left[\left(\sum_{r=0}^{\le\frac{1}{2}n} A_{nr}\right)^2 + \left(\sum_{r=0}^{\le\frac{1}{2}(n-1)} B_{nr}\right)^2\right] \tag{3-17}$$

Similar equations can be derived for the other quantities. The results are:

$$J_{(\lambda+2)+\frac{1}{2}} J_{\lambda+\frac{1}{2}} + J_{-(\lambda+2)-\frac{1}{2}} J_{-\lambda-\frac{1}{2}} = -\left(\frac{2}{\pi z}\right)\left[\sum_{r=0}^{\le\frac{1}{2}(\lambda+2)} \sum_{r'=0}^{\le\frac{1}{2}\lambda} A_{\lambda+2,r} A_{\lambda,r'} + \sum_{r=0}^{\le\frac{1}{2}(\lambda+1)} \sum_{r'=0}^{\le\frac{1}{2}(\lambda-1)} B_{\lambda+2,r} B_{\lambda,r'}\right] \tag{3-18}$$

$$J_{-\lambda-\frac{1}{2}} J_{-(\lambda-1)-\frac{1}{2}} + J_{\lambda+\frac{1}{2}} J_{(\lambda+1)+\frac{1}{2}} = \left(\frac{2}{\pi z}\right)\left[\sum_{r=0}^{\le\frac{1}{2}\lambda} \sum_{r'=0}^{\le\frac{1}{2}(\lambda-1)} A_{\lambda,r} B_{\lambda-1,r'} - \sum_{r=0}^{\le\frac{1}{2}(\lambda-1)} \sum_{r'=0}^{\le\frac{1}{2}(\lambda-1)} A_{\lambda-1,r} B_{\lambda,r'}\right]$$

$$\lambda \ne 0 \tag{3-19}$$

It is thus possible to express the trigonometric functions of η_λ and the derivatives of η_λ simply in terms of ratios of polynomials. These functions could then be substituted into equations 1-3 through 1-9 to give the cross section and the moments of the cross section in terms of polynomials of z.

SECTION IV

EXPANSION OF $\frac{1}{\gamma}$ AND $\frac{1}{\eta}$ IN POWER SERIES AND NUMERICAL CALCULATIONS OF $\frac{1}{\gamma}$

In this section we transform Eqs. 2-18, 2-26, and 2-37 to forms which may be evaluated numerically. The above equations are considered in order.

To evaluate Eq. 2-18 we need an expression for ρ_ℓ . Hence we need an expression for $\rho\left(x\frac{T}{2\theta}\right)$ (Eq. 2-13). From Eqs. 3-15 and 3-17 we obtain the result

$$\eta'_\lambda \bar{\eta}'_{\lambda+1} + \bar{\eta}'_\lambda \eta'_{\lambda+1} = \frac{\left(\frac{2}{\pi z}\right)\left(\frac{2}{\pi \bar{z}}\right)}{\left[J^2_{-\lambda-\frac{1}{2}} + J^2_{\lambda+\frac{1}{2}}\right]\left[\bar{J}^2_{-(\lambda+1)-\frac{1}{2}} + \bar{J}^2_{(\lambda+1)+\frac{1}{2}}\right]} + \frac{\left(\frac{2}{\pi \bar{z}}\right)\left(\frac{2}{\pi z}\right)}{\left[\bar{J}^2_{-\lambda-\frac{1}{2}} + \bar{J}^2_{\lambda+\frac{1}{2}}\right]\left[J^2_{-(\lambda+1)-\frac{1}{2}} + J^2_{(\lambda+1)+\frac{1}{2}}\right]}$$

We now consider the sum over λ with the aid of Eq. 3-17:

For $\lambda = 0$

$$\eta'_0 \bar{\eta}'_1 + \bar{\eta}'_0 \eta'_1 = \frac{\bar{z}^2}{\bar{z}^2+1} + \frac{z^2}{z^2+1} \tag{4-1}$$

For $\lambda = 1$

$$\eta'_1 \bar{\eta}'_2 + \bar{\eta}'_1 \eta'_2 = \frac{z^2}{z^2+1}\,\frac{\bar{z}^4}{\bar{z}^4+3\bar{z}^2+9} + \frac{\bar{z}^2}{\bar{z}^2+1}\,\frac{z^4}{z^4+3z^2+9} \tag{4-2}$$

We truncate the sum at this point. By expanding the denominators in power series we find that (for $z < 1$)

$$\sum_{\lambda}(\lambda+1)\left(\eta'_\lambda\bar{\eta}'_{\lambda+1}+\bar{\eta}'_\lambda\eta'_{\lambda+1}\right)=\left(z^2-z^4+z^6-z^8+\cdots\right)+\left(\bar{z}^2-\bar{z}^4+\bar{z}^6-\bar{z}^8+\cdots\right)$$

$$+\frac{2}{9}\left(z^2\bar{z}^4-z^4\bar{z}^4-\frac{1}{3}z^2\bar{z}^6+\cdots\right)+\frac{2}{9}\left(\bar{z}^2z^4-z^4\bar{z}^4-\frac{1}{3}\bar{z}^2z^6+\cdots\right)+\cdots\tag{4-3}$$

We now change variables from $\bar{z}$ to $\bar{E}^*$ where

$$\bar{E}^*=\lambda^2\bar{z}^2=\bar{\gamma}^2=E^*+X\tag{4-4}$$

and eliminate E^* to obtain an expression in terms of $\bar{E}^*$ and X . Thus

$$\sum_{\lambda}(\lambda+1)\left(\eta'_\lambda\bar{\eta}'_{\lambda+1}+\bar{\eta}'_\lambda\eta'_{\lambda+1}\right)=\frac{1}{\lambda^2}\left(2\bar{E}^*-X\right)$$

$$-\frac{1}{\lambda^4}\left(2\bar{E}^{*2}-2X\bar{E}^*+X^2\right)$$

$$+\frac{1}{\lambda^6}\left(\frac{22}{9}\bar{E}^{*3}-\frac{33}{9}X\bar{E}^{*2}+\frac{29}{9}X^2\bar{E}^*-X^3\right)$$

$$-\frac{1}{\lambda^8}\left(\frac{70}{27}\bar{E}^{*4}-\frac{140}{27}X\bar{E}^{*3}+\frac{180}{27}X^2\bar{E}^{*2}-\frac{110}{27}X^3\bar{E}^*+X^4\right)\tag{4-5}$$

In terms of $\bar{E}^*$, the integral in Eq. 2-13 becomes

$$\frac{1}{2}\int \sqrt{\bar{E}^*-x}\ \bar{E}^{*\frac{1}{2}} \sum_{\lambda} (\lambda+1)(\eta'_\lambda \bar{\eta}'_{\lambda+1} + \bar{\eta}'_\lambda \eta'_{\lambda+1})\ e^{-\bar{E}^*} d\bar{E}^* \tag{4-6}$$

Thus, Eq. 2-13 becomes

$$\begin{aligned} s\left(x\frac{T}{2\theta}\right) = \frac{x^2}{2}\int \sqrt{\bar{E}^*-x}\ \bar{E}^{*\frac{1}{2}} \Big[& \frac{1}{\lambda^2}(2\bar{E}^*-x) - \frac{1}{\lambda^4}(2\bar{E}^{*2}-2x\bar{E}^*+x^2) \\ & + \frac{1}{\lambda^6}\left(\frac{22}{9}\bar{E}^{*3} - \frac{33}{9}x\bar{E}^{*2} + \frac{29}{9}x^2\bar{E}^* - x^3\right) \\ & - \frac{1}{\lambda^8}\left(\frac{70}{27}\bar{E}^{*4} - \frac{140}{27}x\bar{E}^{*3} + \frac{180}{27}x^2\bar{E}^{*2} - \frac{110}{27}x^3\bar{E}^* + x^4\right) + \cdots \Big] e^{-\bar{E}^*} d\bar{E}^* \end{aligned} \tag{4-7}$$

The series in the above integral is convergent only when $Z < 1$ and $\bar{Z} < 1$ (Eq. 4-3) and the limits on the integral are zero and infinity. The integrand, however, contains $e^{-\bar{E}^*}$ as a factor. Since $\bar{E}^* = \lambda^2 \bar{Z}^2$ this factor, for large values of λ^2, considerably dampens the contribution of large values of $\bar{Z}$ to the integral. The result of integrating the divergent series is then probably an asymptotic series. Thus from the above result it follows that

$$s\left(x\frac{T}{2\theta}\right) = \frac{1}{2}\left[\frac{1}{\lambda^2}s_2\left(x\frac{T}{2\theta}\right) - \frac{1}{\lambda^4}s_4\left(x\frac{T}{2\theta}\right) + \frac{1}{\lambda^6}s_6\left(x\frac{T}{2\theta}\right) - \frac{1}{\lambda^8}s_8\left(x\frac{T}{2\theta}\right)\right] \tag{4-8}$$

where

$$s_2\left(x\frac{T}{2\theta}\right) = 2x^2F_1 - x^3F_0$$

$$s_4\left(x\frac{T}{2\theta}\right) = 2x^2F_2 - 2x^3F_1 + x^4F_0$$

$$s_6\left(x\frac{T}{2\theta}\right) = \frac{22}{9}x^2F_3 - \frac{33}{9}x^3F_2 + \frac{24}{9}x^4F_1 - x^5F_0$$

$$s_8\left(x\frac{T}{2\theta}\right) = \frac{20}{27}x^2F_4 - \frac{140}{27}x^3F_3 + \frac{180}{27}x^4F_2 - \frac{110}{27}x^5F_1 + x^6F_0 \tag{4-9}$$

and

$$F_n = \int\sqrt{E^*-x}\;E^{*\,n+\frac{1}{2}}\,e^{-E^*}\,dE^* \tag{4-10}$$

Thus $s\left(x\frac{T}{2\theta}\right)$ is expressed as a power series in $\frac{1}{\lambda^2}$. If we define

$$s_{n\ell} = \frac{-\ell\, s_n(-\ell) + (\ell+1)\, s_n(\ell+1)}{2\ell+1} \tag{4-11}$$

then from Eq. 2-14 we find that

$$s_\ell = \frac{1}{2}\left[\frac{1}{\lambda^2}s_{2\ell} - \frac{1}{\lambda^4}s_{4\ell} + \frac{1}{\lambda^6}s_{6\ell} - \frac{1}{\lambda^8}s_{8\ell} + \cdots\right] \tag{4-12}$$

Equation 2-18 thus becomes

$$\frac{1}{\gamma} = \frac{8\pi n\sigma^2}{3}\sqrt{\frac{KT}{\pi m}}\left(\frac{\delta}{\sigma}\right)^2 \frac{\sum_\ell (2\ell+1)e^{-\epsilon_\ell}\left(\frac{1}{\lambda^2}s_{2\ell} - \frac{1}{\lambda^4}s_{4\ell} + \frac{1}{\lambda^6}s_{6\ell} - \frac{1}{\lambda^8}s_{8\ell} + \cdots\right)}{\sum_\ell (2\ell+1)e^{-\epsilon_\ell}} \tag{4-13}$$

If we define

$$S_n = \frac{\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell} s_{n\ell}}{\sum_\ell (2\ell+1)\, e^{-\epsilon_\ell}} \tag{4-14}$$

so

$$\frac{1}{\tau} = \frac{8\pi n\sigma^2}{3}\sqrt{\frac{kT}{\pi m}}\left(\frac{\delta}{\sigma}\right)^2\left[\frac{1}{\lambda^2}S_2 - \frac{1}{\lambda^4}S_4 + \frac{1}{\lambda^6}S_6 - \frac{1}{\lambda^8}S_8 + \cdots\right] \tag{4-15}$$

We are now in a position to evaluate $\frac{1}{\tau}$ numerically. S_n depends upon the quantity X , which in turn depends upon ℓ and $\frac{\theta}{T}$. In the calculations, a value for $\frac{\theta}{T}$ is chosen and the four S_n values computed. The quantity

$$S = \frac{1}{\lambda^2}S_2 - \frac{1}{\lambda^4}S_4 + \frac{1}{\lambda^6}S_6 - \frac{1}{\lambda^8}S_8 + \cdots$$

is then computed. If the series is asymptotic the absolute error in S will be less than the last term retained in the series. Values of S_n for nine values of $\frac{\theta}{T}$ are given in Table 4-1. The classical limit for $\frac{1}{\tau}$ is

$$\frac{1}{\tau} = \frac{16\pi n\sigma^2}{3}\sqrt{\frac{kT}{\pi m}}\left(\frac{m\sigma^2}{T}\right)\left(\frac{\delta}{\sigma}\right)^2$$

Therefore, the quantity $\frac{\lambda^2}{4}\frac{T}{\theta}S$ is the reciprocal of the ratio of the relaxation time to the value in the classical limit. A plot of $\frac{\lambda^2}{4}\frac{T}{\theta}S$ against $\frac{1}{\lambda^2}$ for $\frac{\theta}{T}$ = 2, 1, .5, .25, .125, .0625, and .015625 is given in Fig. 1.

Table 4-1

$\frac{\Theta}{T}$	5	2	1	.5	.25
S_2	.3582	8.386	8.446	5.605	3.314
S_4	3.803	43.80	34.11	20.47	11.19
S_6	46.06	293.4	194.5	109.3	57.20
S_8	545.9	1976.	1158.	615.0	311.8

$\frac{\Theta}{T}$	.125	.0625	.03125	.015625
S_2	1.814	.9513	.4875	.2468
S_4	5.820	2.959	1.491	.7478
S_6	29.06	14.61	7.322	3.664
S_8	156.1	77.95	38.94	19.46

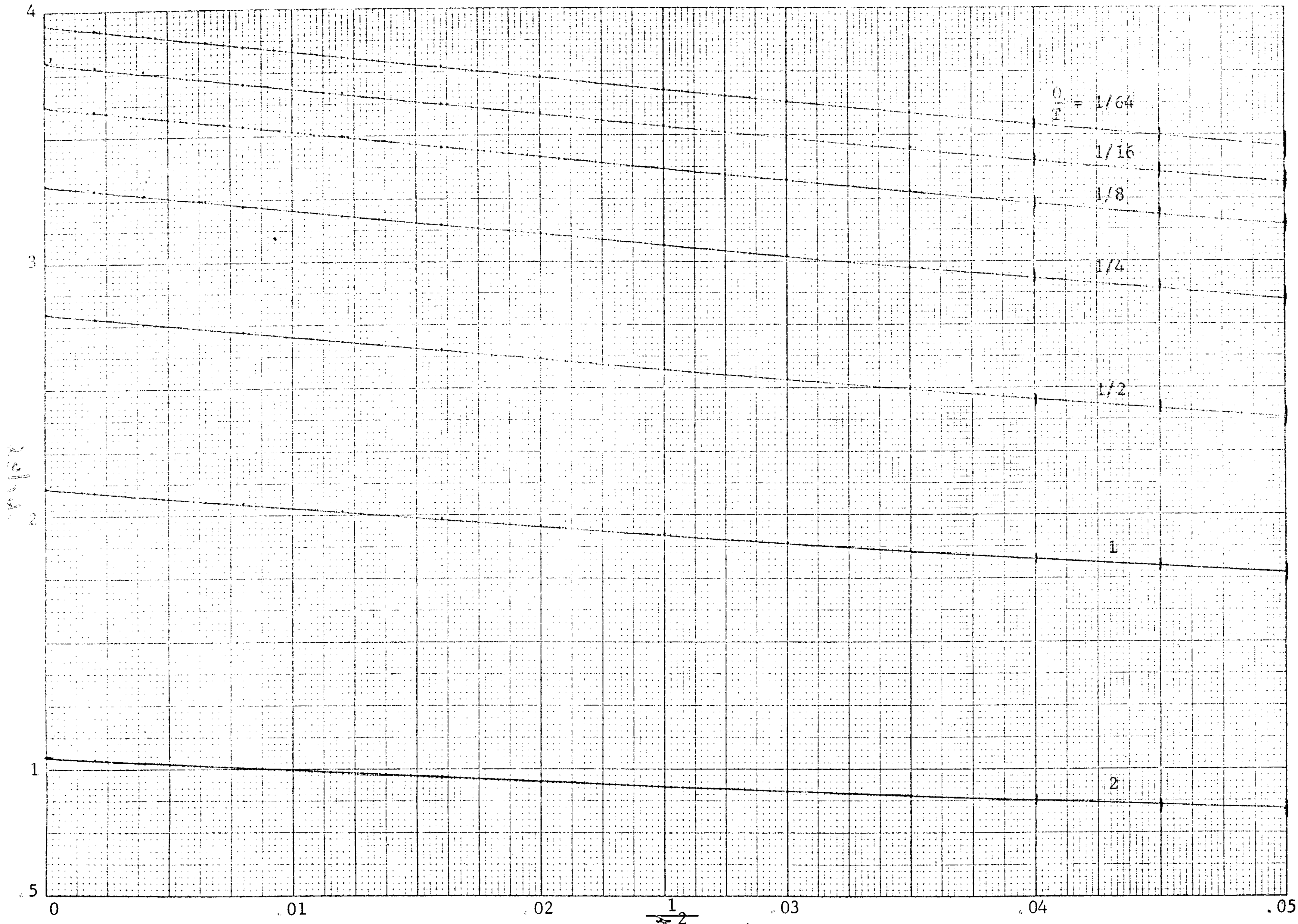

Figure 1

We now turn our attention to Eqs. 2-26. In order to compare this with Uehling's result[2] on the rigid sphere we must consider the effect of Bose-Einstein statistics. Formally the expansion is obtained from that for Boltzmann statistics by summing only over even values of λ and multiplying by 2. Eq. 2-26 thus becomes, in the Bose Einstein case,

$$\frac{1}{\eta_0} = \frac{64\pi}{5\sqrt{\pi m k T}} \sum_{\lambda(\text{even})} \frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)} \int \frac{\gamma^7}{\mathcal{H}^2} \sin^2(\eta_{\lambda+2}-\eta_\lambda) e^{-\gamma^2} d\gamma \tag{4-16}$$

We now transform variables from γ to Z (Eq. 2-10).

$$\frac{1}{\eta_0} = \frac{64\pi}{5\sqrt{\pi m k T}} \sum_{\lambda(\text{even})} \frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)} \lambda^8 \sigma^2$$

$$\int Z^5 \sin^2(\eta_{\lambda+2}-\eta_\lambda) e^{-\lambda^2 Z^2} dZ \tag{4-17}$$

In the sum over λ we tak only the $\lambda=0$ term. Eq. 3-7 gives $\sin^2(\eta_{\lambda+2}-\eta_\lambda)$ and Eq. 3-17 allows 3-7 to be evaluated in terms of Z. The result is

$$\sin^2(\eta_2-\eta_0) = \frac{9}{Z^2} \frac{1}{1+\frac{3}{Z^2}+\frac{9}{Z^4}} \tag{4-18}$$

Thus Eq. 4-17 becomes

$$\frac{1}{\eta_0} = \frac{6\cdot 64\pi\sigma^2}{5\sqrt{\pi m k T}} \lambda^8 \int \frac{Z^3}{1+\frac{3}{Z^2}+\frac{9}{Z^4}} e^{-\lambda^2 Z^2} dZ \tag{4-19}$$

The ratio of polynomials in the integrand is expanded in a power series. This gives

$$\frac{1}{\eta_0} = \frac{128\pi\sigma^2}{15\sqrt{\pi m KT}} \lambda^8 \int \left(z^7 - \tfrac{1}{3}z^9 + \tfrac{1}{27}z^{13} + \cdots\right) e^{-\lambda^2 z^2} dz \qquad (4\text{-}20)$$

But

$$\int z^{2n+1} e^{-a^2 z^2} dz = \frac{1}{2}\frac{n!}{a^{2n+2}} \qquad (4\text{-}21)$$

so

$$\frac{1}{\eta_0} = \frac{64\pi\sigma^2}{15\sqrt{\pi m KT}} \lambda^8 \left[\frac{3!}{\lambda^8} - \frac{1}{3}\frac{4!}{\lambda^{10}} + \frac{1}{27}\frac{6!}{\lambda^{14}}\right] \qquad (4\text{-}22)$$

or

$$\frac{1}{\eta_0} = \frac{128\pi\sigma^2}{5\sqrt{\pi m KT}} \left[1 - \frac{4}{3}\frac{1}{\lambda^2} + \frac{40}{9}\frac{1}{\lambda^6} + \cdots\right] \qquad (4\text{-}23)$$

The first term in this expansion agrees with the result obtained by Uehling. It can be pointed out that if another term in the sum over λ, ($\lambda = 2$), is considered and a similar procedure carried out, the largest correction term is of the order of $\frac{1}{\lambda^8}$.

For Fermi-Dirac statistics we sum Eq. 2-26 over odd values of λ and multiply by 2. This gives

$$\frac{1}{\eta_0} = \frac{64\pi\sigma^2}{5\sqrt{\pi m KT}} \sum_{\lambda(\mathrm{odd})} \frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)} \lambda^8 \int z^5 \sin^2(\eta_{\lambda+2} - \eta_\lambda) e^{-\lambda^2 z^2} dz \qquad (4\text{-}24)$$

In summing over λ we take only the $\lambda=1$ term. Again we use Eq. 3-7 and Eq. 3-17 to evaluate $\sin^2(\eta_{\lambda+2}-\eta_\lambda)$

$$\sin^2(\eta_3-\eta_1)=\frac{25}{z^2}\,\frac{1}{1+\frac{7}{z^2}+\frac{51}{z^4}+\frac{270}{z^6}+\frac{225}{z^8}} \tag{4-25}$$

Eq. 4-24 becomes

$$\frac{1}{\eta_0}=\frac{6.64\pi\sigma^2}{\sqrt{\pi m k T}}\,\bar{\lambda}^8\int\frac{z^3\,e^{-\bar{\lambda}^2 z^2}}{1+\frac{7}{z^2}+\frac{51}{z^4}+\frac{270}{z^6}+\frac{225}{z^8}}\,dz$$

We expand the ratio of polynomials,

$$\frac{1}{\eta_0}=\frac{6.64\pi\sigma^2}{\sqrt{\pi m k T}}\,\bar{\lambda}^8\,\frac{1}{225}\int\left(z^{11}-\frac{6}{5}z^{13}+\frac{273}{225}z^{15}-\frac{1367}{1125}z^{17}\right)e^{-\bar{\lambda}^2 z^2}dz \tag{4-26}$$

and integrate using Eq. 4-21 to find that

$$\frac{1}{\eta_0}=\frac{128\pi\sigma^2}{5\sqrt{\pi m k T}}\left[4\,\frac{1}{\bar{\lambda}^4}-\frac{144}{5}\,\frac{1}{\bar{\lambda}^6}+\frac{15288}{75}\,\frac{1}{\bar{\lambda}^8}-\frac{3674496}{125}\,\frac{1}{\bar{\lambda}^{10}}\right] \tag{4-27}$$

In this case the first correction term is of order $\frac{1}{\bar{\lambda}^{12}}$.

We now turn to the evaluation of $\frac{1}{\eta_2}$ (Eq. 2-37), and consider only the case of Boltzmann statistics. In the evaluation of the first term (Eq. 2-29), we retain only the $\lambda=0$ and $\lambda=1$ terms in the sum over λ. From Eq. 1-7 we obtain

$$Q^{(2)}_{\mathrm{el},A+B+C_1}(\bar{\lambda}^{q}\bar{\lambda}^{b}\bar{\lambda}^{q}\bar{\lambda}^{b}) = -\frac{8\pi}{3}\frac{\sigma}{\kappa}\sum_{\lambda}\frac{(\lambda+1)(\lambda+2)}{(2\lambda+3)}(\eta'_{\lambda+2}-\eta'_{\lambda})\sin 2(\eta_{\lambda+2}-\eta_{\lambda})$$

For $\lambda=0$

$$Q^{(2)}_{\mathrm{el},A+B+C_1}(\bar{\lambda}^{q}\bar{\lambda}^{b}\bar{\lambda}^{q}\bar{\lambda}^{b}) = -\frac{16\pi\sigma^2}{9}\frac{1}{Z}(\eta'_2-\eta'_0)\sin 2(\eta_2-\eta_0)$$

This can be evaluated from Eq. 3-10 and 3-15

$$Q^{(2)}_{\mathrm{el},A+B+C_1}(\bar{\lambda}^{q}\bar{\lambda}^{b}\bar{\lambda}^{q}\bar{\lambda}^{b}) = -32\pi\sigma^2\frac{9-Z^4}{(Z^4+3Z^2+9)^2}$$

Thus this term in Eq. 2-29 becomes, after transforming the integration variable to Z :

$$-\frac{256\pi\sigma^2}{5\sqrt{\pi m k T}}\chi^8\int Z^7\frac{9-Z^4}{(Z^4+3Z^2+9)^2}e^{-\chi^2 Z^2}dZ$$

We expand the polynomial ratio and obtain

$$-\frac{256\pi\sigma^2}{5\sqrt{\pi m k T}}\frac{\chi^8}{27}\int\left(3Z^7-2Z^9+\frac{4}{9}Z^{13}\right)e^{-\chi^2 Z^2}dZ$$

Integrating with the aid of Eq. 4-21 gives:

$$-\frac{256\pi\sigma^2}{15\sqrt{\pi m k T}}\left[1-\frac{8}{3}\frac{1}{\chi^2}+\frac{160}{9}\frac{1}{\chi^6}\right]$$

In a similar manner it is found that for $\lambda=1$, Eq, 2-29 becomes

$$-\frac{256\pi\sigma^2}{15\sqrt{\pi m k T}}\left[12\,\frac{1}{\lambda^4}-\frac{576}{5}\,\frac{1}{\lambda^6}\right]$$

Combining these last two equations gives the first term of $\frac{1}{\eta_2}$ through sixth order in $\frac{1}{\lambda}$.

$$-\frac{256\pi\sigma^2}{15\sqrt{\pi m k T}}\left[1-\frac{8}{3}\,\frac{1}{\lambda^2}+12\,\frac{1}{\lambda^4}-\frac{5984}{45}\,\frac{1}{\lambda^6}\right] \tag{4-28}$$

We now turn to the second term in $\frac{1}{\eta_2}$ (Eq. 2-36). To evaluate this we need an expression for $\gamma(x\frac{I}{2\theta})$ (Eq. 2-32). This requires expressions for $\tilde{g}^{(2)}_{el,C_2}(\bar{1}^{4}\bar{1}^{6}1^{4}1^{6})_2$, $\tilde{Q}^{(2)}_{inel}(\bar{1}^{4}\bar{1}^{6}1^{4}1^{6})_2$, and $\tilde{I}_{inel}(\bar{1}^{4}\bar{1}^{6}1^{4}1^{6}|2)_2$. We take each of these in turn and expand them in a power series in z and sum over λ at the same time. Care must be taken at this point to retain all terms of the required order. For $\tilde{g}^{(2)}_{el,C_2}(\bar{1}^{4}\bar{1}^{6}1^{4}1^{6})_2$ we retain parts of the $\lambda=2$ term. In each case we use Eqs. 3-7 through 3-16 to reduce the above quantities to expressions involving Bessel functions. Then Eqs. 3-17 through 3-19 are used to reduce the Bessel functions to ratios of polynomials which are then expanded. The results are

$$\tilde{g}^{(2)}_{el,C_2}(\bar{1}^{4}\bar{1}^{6}1^{4}1^{6})_2=\frac{16\pi\sigma^2}{9}\left[1-\frac{2}{3}\bar{z}^2-z^2+\frac{2}{3}z^2\bar{z}^2-z^3\bar{z}+\frac{1}{15}\bar{z}^4+z^4\right] \tag{4-29}$$

$$\tilde{Q}^{(2)}_{inel}(\bar{\lambda}^{4}\bar{\lambda}^{6}\lambda^{4}\lambda^{6})_2 = \frac{8\pi\sigma^2}{9}\,\frac{Z}{\bar{Z}}\left[(Z^2+\bar{Z}^2)-(Z^4+\bar{Z}^4)-\frac{8}{15}\bar{Z}^2Z^2\right] \quad (4\text{-}30)$$

$$\tilde{I}_{inel}(\bar{\lambda}^{4}\bar{\lambda}^{6}\lambda^{4}\lambda^{6}|2)_2 = \frac{4\sigma^2}{27}\,Z^3\bar{Z}\left[6-5(Z^2+\bar{Z}^2)\right] \quad (4\text{-}31)$$

Before substituting these three equations into $r(x\frac{I}{2\theta})$, we change the integration variable in Eq. 2-32 from $\bar{\gamma}$ to $\bar{Z}$,

$$r(x\tfrac{I}{2\theta}) = \lambda^8\iint\Big[\tilde{g}^{(2)}_{4,c_2}(\bar{\lambda}^{4}\bar{\lambda}^{6}\lambda^{4}\lambda^{6})_2\,\bar{Z}^7+\tilde{Q}^{(2)}_{inel}(\bar{\lambda}^{4}\bar{\lambda}^{6}\lambda^{4}\lambda^{6})_2\,\bar{Z}^7$$
$$+\frac{4\pi}{15}x^2\,\frac{1}{\lambda^4}\,\tilde{I}_{inel}(\bar{\lambda}^{4}\bar{\lambda}^{6}\lambda^{4}\lambda^{6}|2)_2\,\bar{Z}^3\Big]e^{-\lambda^2\bar{Z}^2}\,d\bar{Z} \quad (4\text{-}32)$$

We now substitute Eqs. 4-29, 4-30 and 4-31 into Eq. 4-32. We change the variable of integration from $\bar{Z}$ to $\bar{E}^*$ where

$$\bar{E}^* = \lambda^2\bar{Z}^2$$

The result is

$$r(x\tfrac{I}{2\theta}) = \frac{1}{2}\left[r_0(x\tfrac{I}{2\theta})+\frac{1}{\lambda^2}r_2(x\tfrac{I}{2\theta})+\frac{1}{\lambda^4}r_4(x\tfrac{I}{2\theta})+\cdots\right] \quad (4\text{-}33)$$

where

$$r_0(x\tfrac{I}{2\theta}) = \frac{388}{9}\pi\sigma^2$$

$$r_2\left(x\frac{T}{2\theta}\right) = -\frac{32\pi\sigma^2}{3}\left(\frac{2\theta}{3}-x\right) + \frac{8\pi\sigma^2}{9}\left[2F_3 - xF_2\right]$$

$$r_4\left(x\frac{T}{2\theta}\right) = \frac{8\pi\sigma^2}{9}\left[336 - 128x + 12x^2\right]$$

$$-\frac{8\pi\sigma^2}{9}\left[\frac{52}{15}F_4 - \frac{52}{15}F_3 + \frac{11}{15}F_2 + \frac{2}{45}F_1\right] \qquad (4\text{-}34)$$

If we define

$$r_{n\ell} = \frac{-\ell\, r(-\ell) + (\ell+1)\, r(\ell+1)}{2\ell+1} \qquad (4\text{-}35)$$

then Eq. 2-33 becomes

$$r_\ell = \frac{1}{2}\left[r_{0\ell} + \frac{1}{\lambda^2}r_{2\ell} + \frac{1}{\lambda^4}r_{4\ell} + \cdots\right] \qquad (4\text{-}36)$$

and Eq. 2-36 becomes

$$\frac{8}{5\sqrt{\pi m k T}}\;\frac{\sum_\ell (2\ell+1)\,e^{-\epsilon_\ell}\left(r_{0\ell} + \frac{1}{\lambda^2}r_{2\ell} + \frac{1}{\lambda^4}r_{4\ell} + \cdots\right)}{\sum_\ell (2\ell+1)\,e^{-\epsilon_\ell}} \qquad (4\text{-}37)$$

or

$$\frac{8}{5\sqrt{\pi m k T}}\left[R_0 + \frac{1}{\lambda^2}R_2 + \frac{1}{\lambda^4}R_4 + \cdots\right] \qquad (4\text{-}38)$$

where

$$R_n = \frac{\sum_l (2l+1) e^{-\epsilon_l} r_{nl}}{\sum_l (2l+1) e^{-\epsilon_l}} \qquad (4\text{-}39)$$

The R_n may be evaluated numerically.

APPENDIX I

Certain combinations of Bessel functions appear quite frequently in evaluating the cross section and its moments. These are listed below for small values of λ ,

$$J_{-\frac{1}{2}}^2 + J_{\frac{1}{2}}^2 = \frac{2}{\pi z}$$

$$J_{-\frac{3}{2}}^2 + J_{\frac{3}{2}}^2 = \frac{2}{\pi z}\left(1 + \frac{1}{z^2}\right)$$

$$J_{-\frac{5}{2}}^2 + J_{\frac{5}{2}}^2 = \frac{2}{\pi z}\left(1 + \frac{3}{z^2} + \frac{9}{z^4}\right)$$

$$J_{-\frac{7}{2}}^2 + J_{\frac{7}{2}}^2 = \frac{2}{\pi z}\left(1 + \frac{6}{z^2} + \frac{45}{z^4} + \frac{225}{z^6}\right)$$

$$J_{-\frac{9}{2}}^2 + J_{\frac{9}{2}}^2 = \frac{2}{\pi z}\left(1 + \frac{10}{z^2} + \frac{135}{z^4} + \frac{1575}{z^6} + \frac{11025}{z^8}\right)$$

$$J_{\frac{1}{2}} J_{\frac{5}{2}} + J_{-\frac{1}{2}} J_{-\frac{5}{2}} = \frac{2}{\pi z}\left(\frac{1}{z^2}\right)\left(3 - z^2\right)$$

$$J_{\frac{3}{2}} J_{\frac{7}{2}} + J_{-\frac{3}{2}} J_{-\frac{7}{2}} = \frac{2}{\pi z}\left(\frac{1}{z^4}\right)\left(15 + 9z^2 - z^4\right)$$

$$J_{\frac{5}{2}} J_{\frac{9}{2}} + J_{-\frac{5}{2}} J_{-\frac{9}{2}} = \frac{2}{\pi z}\left(\frac{1}{z^6}\right)\left(315 + 75z^2 + 18z^4 - z^6\right)$$

$$J_{-\frac{3}{2}} J_{-\frac{1}{2}} - J_{\frac{3}{2}} J_{\frac{1}{2}} = -\frac{2}{\pi z}\left(\frac{1}{z}\right)$$

$$J_{-\frac{5}{2}} J_{-\frac{3}{2}} - J_{\frac{5}{2}} J_{\frac{3}{2}} = -\frac{2}{\pi z}\left(\frac{1}{z^3}\right)\left(3+2z^2\right)$$

$$J_{-\frac{7}{2}} J_{-\frac{5}{2}} - J_{\frac{7}{2}} J_{\frac{5}{2}} = -\frac{2}{\pi z}\left(\frac{1}{z^5}\right)\left(45+12z^2+3z^4\right)$$

$$J_{-\frac{9}{2}} J_{-\frac{7}{2}} - J_{\frac{9}{2}} J_{\frac{7}{2}} = -\frac{2}{\pi z}\left(\frac{1}{z^7}\right)\left(1575+270z^2+30z^4+4z^6\right)$$

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